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STABILITY RESULT FOR DYNAMIC INVERSION DEVISED TO CONTROL LARGE FLEXIBLE AIRCRAFT

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Abstract

High performance aircraft of the future will be designed lighter, more maneuverable, and operate over an ever expanding flight envelope. One of the largest differences from the flight control perspective between current and future advanced aircraft is elasticity. Over the last decade, dynamic inversion methodology has gained considerable popularity in application to highly maneuverable fighter aircraft, which were treated as rigid vehicles.

This paper is an initial attempt to establish global stability results for dynamic inversion methodology as applied to a large, flexible aircraft. This work builds on a previous result for rigid fighter aircraft and adds a new level of complexity that is the flexible aircraft dynamics, which cannot be ignored even in the most basic flight control. The results arise from observations of the control laws designed for a new generation of the High-Speed Civil Transport aircraft.

Introduction

The advanced aircraft under consideration is a next generation supersonic transport aircraft, which due to aerodynamic factors will be long and slender and because of economics must be as light as possible. These factors contribute to making this aircraft very flexible with the first few elastic modes lying within the bandwidth of the traditional flight control system.

Over the last decade, dynamic inversion methodology has gained considerable popularity in application to highly maneuverable fighter aircraft[1] which makes it a candidate that might benefit the advanced highly flexible aircraft of the future. This methodology has been modified and successfully applied to a new generation High-Speed Civil Transport vehicle[2,3]. However, there is very little applicable theory available on the stability of dynamic inversion based closed loop systems. Some recent work has been done by Morton, et. al.[4] focusing on the global stability of a rigid fighter aircraft.

This paper is an initial attempt to establish stability results for dynamic inversion methodology as applied to a large, flexible vehicle. The method of reasoning follows the work of Morton, et. al.. This work builds on the results in reference 4 and adds a new level of complexity that is the flexible aircraft dynamics, which cannot be ignored even in the most basic flight control. The results arise from observations of the control laws designed for a new generation of the High-Speed Civil Transport aircraft that were described in some detail in references 2 and 3.

In order to get initial results, the problem has been simplified for analytical work while retaining the essential characteristics. The simplification involved considering longitudinal dynamics with a single elastic mode and a control law designed for flight control only. In addition, throughout this paper the analysis considers the inner loop of the dynamic inversion only, i.e., the $\dot{y}^{des}$ to y portion. It is important to note that the nature of $\dot{y}^{des}$ impacts the overall closed loop dynamics but will not be discussed here.

The paper is organized as follows. The first section describes the equations of motion for this class of vehicles, followed by a definition of equilibrium set in section two. Section three presents the dynamic inversion results. Section four discusses stability results obtained to date, followed by concluding remarks.

System Equations of Motion

The equations of motion for an HSCT class vehicles (illustrated in figure 1) are rather involved (see reference 3). They contain the usual aerodynamic forces and moments acting on the so called rigid vehicle and the actuator dynamics. In addition, equations for flexible body dynamics, their interaction with the traditional aerodynamics, the interactions among themselves, and influence on the control surface activity are all also included. In this paper we limit our discussion to standard rigid body dynamics, one flexible mode, and interaction between the two. In addition, the control surfaces have been combined into one variable. Hence, a somewhat simplified longitudinal equations of motion for an HSCT type aircraft are given below

$$
\begin{aligned}
\begin{bmatrix}\dot{u}\\ \dot{w}\\ \dot{q}\\ \dot{\theta}\\ \ddot{\eta}\end{bmatrix} =& \begin{pmatrix}-qw\\ qu\\ 0\\ q\\ 0\end{pmatrix} + \begin{pmatrix}0\\ 0\\ 0\\ 0\\ -2\zeta\omega\dot{\eta}/m_\eta - \omega^2\eta/m_\eta\end{pmatrix} + \begin{pmatrix}-g\sin\theta\\ g\cos\theta\\ 0\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}T/\mathrm{m}\\ 0\\ 0\\ 0\\ 0\end{pmatrix} + \frac{1}{2}\rho S V^2 \left\{ \begin{pmatrix}C_x(\alpha,\mathrm{M})/\mathrm{m}\\ C_z(\alpha,\mathrm{M})/\mathrm{m}\\ \bar{c}C_M(\alpha,\mathrm{M})/I_y\\ 0\\ 0\end{pmatrix} \right. \\
&+ \begin{pmatrix}0\\ C_{z,\eta}(\alpha,\mathrm{M},GW^*)/\mathrm{m}\\ \bar{c}C_{M,\eta}(\alpha,\mathrm{M},GW^*)/I_y\\ 0\\ \bar{q}C_{\eta\eta}(\mathrm{M},GW^*)/m_\eta\end{pmatrix}\eta + \begin{pmatrix}0\\ C_{z,\eta}(\alpha,\mathrm{M},GW^*)/\mathrm{m}\\ \bar{c}C_{M,\eta}(\alpha,\mathrm{M},GW^*)/I_y\\ 0\\ \bar{q}C_{\eta\eta}(\mathrm{M},GW^*)/m_\eta\end{pmatrix}\dot{\eta} \\
&\left. + \frac{1}{m_\eta}\begin{pmatrix}0\\ 0\\ 0\\ 0\\ C_{\eta u}(\mathrm{M},GW^*)u + C_{\eta w}(\mathrm{M},GW^*)w + C_{\eta q}(\mathrm{M},GW^*)q\end{pmatrix} + \begin{pmatrix}C_{x,\delta 1}(\alpha,\mathrm{M},GW^*)/\mathrm{m}\\ C_{z,\delta 1}(\alpha,\mathrm{M},GW^*)/\mathrm{m}\\ \bar{c}C_{M,\delta 1}(\alpha,\mathrm{M},GW^*)/I_y\\ 0\\ C_{\eta\delta 1}(\mathrm{M},GW^*)/m_\eta\end{pmatrix}\delta_1 \right\}
\end{aligned} \tag{1}
$$

where

m = vehicle mass, m_η = generalized mass associated with an elastic mode

GW^* = mass distibution, T = thrust, M = Mach number, V = speed

$\bar{c}$ = mean aerodynamic cord, S = planform area, $\bar{q}$ = dynamic pressure, ρ = density

This paper considers a system with two control effectors, which are a traditionally low-bandwidth thrust and a relatively much faster acting elevator control. Though not explicitly written as a control, mass distribution GW^*, can certainly be considered one. The GW^* can be changed by shifting fuel around the vehicle as is done in the Concorde. In advanced version of a HSCT, there are no plans to employ it as such but GW^* remains a degree of freedom in the system.

The initent is to explore the stability of aircraft undergoing rapid maneuvering, which implies that the behavior of the fast states and parameters affecting them is of primary concern. This allows the treatment of thrust as well as the parametric dependence on GW^* to be considered constant. In other words, the problem is reduced to a SISO control problem. Furthermore, in the subsonic regime we can legitimately drop the coefficient dependence on the Mach number.

Equilibrium Set

The system equations have the form

$$\dot{x} = f(x,\delta)$$

where x is a vector in R^n and δ is a vector in R^m. Let $\overline{U}$ denote a set of allowed control values in R^m. Define the equilibrium set

$$\overline{M} = \left\{(x,\delta) \mid f(x,\delta) = 0,\ \delta \in \overline{U}\right\}.$$

Projecting $\overline{M}$ onto the first factor x we obtain M, the set of equilibrium states.

$$M = \left\{ x : f(x,\delta) = 0,\ \delta \in \bar{U} \right\}$$

Note that $\bar{M}$ and M depend on the specified control limits.

The equilibrium set of a system with m inputs is typically m-dimensional. The system modeled by (1) has three inputs (T, GW^*, δ) and its equilibrium set M is three-dimensional. We choose to use T, θ, and GW^* as the coordinates on M. Note that these are a mixture of controls and inputs. This choice allows us to eliminate the elevator control δ_l from the equilibrium equations and leave them in terms of states that will later be shown to comprise internal dynamics of the closed loop dynamic inversion. In addition, choosing GW^* allows for a unique η for a given flight condition.

Suppose that the triplet (T, θ, GW^*) is fixed, then at equilibrium rates and accelerations are zero giving us

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{pmatrix} -g\sin\theta_o \\ g\cos\theta_o \\ 0 \\ 0 \\ -\omega^2\eta / m_\eta \end{pmatrix} + \begin{pmatrix} T_o / \mathrm{m} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2}\rho S V^2 \left\{ \begin{pmatrix} C_x(\alpha)/\mathrm{m} \\ C_z(\alpha)/\mathrm{m} \\ \bar{c}C_M(\alpha)/I_y \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ C_{z,\eta}(\alpha, GW_o^*)/\mathrm{m} \\ \bar{c}C_{M,\eta}(\alpha, GW_o^*)/I_y \\ 0 \\ \bar{q}C_{\eta\eta}\left(GW_o^*\right)/m_\eta \end{pmatrix}\eta \right.$$
$$\left. + \frac{1}{m_\eta}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ C_{\eta u}\left(GW_o^*\right)u + C_{\eta w}\left(GW_o^*\right)w \end{pmatrix} + \begin{pmatrix} C_{x,\delta 1}(\alpha, GW_o^*)/\mathrm{m} \\ C_{z,\delta 1}(\alpha, GW_o^*)/\mathrm{m} \\ \bar{c}C_{M,\delta 1}(\alpha, GW_o^*)/I_y \\ 0 \\ C_{\eta\delta 1}\left(GW_o^*\right)/m_\eta \end{pmatrix}\bar{\delta}_1 \right\} \tag{2}$$

Solving the pitching moment equation for the control value gives the following relation

$$\bar{\delta}_1 = -\frac{C_M(\alpha) + C_{M,\eta}(\alpha, GW_o^*)\eta}{C_{M,\delta 1}(\alpha, GW_o^*)} \tag{3}$$

Substituting for the controls in (2) and simplifying gives

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -g\sin\theta_o + \frac{T_o}{\mathrm{m}} \\ g\cos\theta_o \\ 0 \\ -\omega^2\eta / m_\eta \end{pmatrix} + \frac{1}{2}\rho S V^2 \left\{ \begin{pmatrix} \frac{C_x(\cdot)}{\mathrm{m}} - \frac{C_{x,\delta 1}(\cdot)}{\mathrm{m}}\frac{C_M(\cdot)}{C_{M,\delta 1}(\cdot)} \\ \frac{C_z(\cdot)}{\mathrm{m}} - \frac{C_{z,\delta 1}(\cdot)}{\mathrm{m}}\frac{C_M(\cdot)}{C_{M,\delta 1}(\cdot)} \\ \frac{\bar{c}}{I_y}C_M(\cdot) - \frac{\bar{c}}{I_y}C_M(\cdot) \\ -\frac{C_{\eta\delta 1}(\cdot)}{m_\eta}\frac{C_M(\cdot)}{C_{M,\delta 1}(\cdot)} \end{pmatrix} + \begin{pmatrix} -\frac{C_{x,\delta 1}(\cdot)}{\mathrm{m}}\frac{C_{M,\eta}(\cdot)}{C_{M,\delta 1}(\cdot)} \\ \frac{C_{z,\eta}(\cdot)}{\mathrm{m}} - \frac{C_{z,\delta 1}(\cdot)}{\mathrm{m}}\frac{C_{M,\eta}(\cdot)}{C_{M,\delta 1}(\cdot)} \\ \frac{\bar{c}}{I_y}C_{M,\eta}(\cdot) - \frac{\bar{c}}{I_y}C_{M,\eta}(\cdot) \\ \frac{\bar{q}}{m_\eta}C_{\eta\eta}(\cdot) - \frac{C_{\eta\delta 1}(\cdot)}{m_\eta}\frac{C_{M,\eta}(\cdot)}{C_{M,\delta 1}(\cdot)} \end{pmatrix}\eta \right.$$
$$\left. + \frac{1}{m_\eta}\begin{pmatrix} 0 \\ 0 \\ 0 \\ C_{\eta u}(\cdot)u + C_{\eta w}(\cdot)w \end{pmatrix} \right\} \tag{4}$$

The pitch rate dynamics disappear by design, which leave velocity and elastic mode dynamics. In the equilibrium, the force due to velocity acting on the elastic dynamics is balanced by the structural and aerodynamic forces and moments. Consequently, the dynamics of (u, w, η) define the rest of the equilibrium.

The dynamics on the u-w-η surface are

$$\begin{pmatrix} \mathrm{mg}\sin\theta_o - T_o \\ -\mathrm{mg}\cos\theta_o \\ 0 \end{pmatrix} = \frac{1}{2}\rho S V^2 \begin{pmatrix} C_x(\alpha) - C_{x,\delta 1}(\alpha, GW_o^*)\dfrac{C_M(\alpha)}{C_{M,\delta 1}(\alpha, GW_o^*)} \\ C_z(\alpha) - C_{z,\delta 1}(\alpha, GW_o^*)\dfrac{C_M(\alpha)}{C_{M,\delta 1}(\alpha, GW_o^*)} \\ V\left(C_{\eta u}(GW_o^*)\cos\alpha + C_{\eta w}(GW_o^*)\sin\alpha\right) - C_{\eta\delta 1}(GW_o^*)\dfrac{C_M(\alpha)}{C_{M,\delta 1}(\alpha, GW_o^*)} \end{pmatrix}$$

$$+\frac{1}{2}\rho S V^2 \begin{pmatrix} -C_{x,\delta 1}(\alpha, GW_o^*)\dfrac{C_{M,\eta}(\alpha, GW_o^*)}{C_{M,\delta 1}(\alpha, GW_o^*)} \\ C_{z,\eta}(\alpha) - C_{z,\delta 1}(\alpha, GW_o^*)\dfrac{C_{M,\eta}(\alpha, GW_o^*)}{C_{M,\delta 1}(\alpha, GW_o^*)} \\ \bar{q}C_{\eta\eta}(GW_o^*) - C_{\eta\delta 1}(GW_o^*)\dfrac{C_{M,\eta}(\alpha, GW_o^*)}{C_{M,\delta 1}(\alpha, GW_o^*)} \end{pmatrix}\eta + \begin{pmatrix} 0 \\ 0 \\ -\omega^2\eta \end{pmatrix} \tag{5}$$

The forces of thrust and gravity must be balanced by the aerodynamic forces and moments. This formulation allows for a natural separation of dynamics due to rigid body and elastic interactions.

We introduce the idea of equilibrium aerodynamic functions that will have direct connection to the aerodynamic forces required to maintain equilibrium state.

Definition: For a system represented by equation (5), the equilibrium aerodynamic functions due to rigid body effects are $\bar{C}_x(\alpha)$, $\bar{C}_z(\alpha)$, and $\bar{C}_\eta(\alpha, V)$ defined by

$$\begin{pmatrix} \bar{C}_x(\alpha) \\ \bar{C}_z(\alpha) \\ \bar{C}_\eta(\alpha, V) \end{pmatrix} = \begin{pmatrix} C_x - C_{x,\delta 1}\dfrac{C_M}{C_{M,\delta 1}}(\alpha) \\ C_z - C_{z,\delta 1}\dfrac{C_M}{C_{M,\delta 1}}(\alpha) \\ V\left(C_{\eta u}\cos\alpha + C_{\eta w}\sin\alpha\right) - C_{\eta\delta 1}\dfrac{C_M(\alpha)}{C_{M,\delta 1}(\alpha)} \end{pmatrix},$$

the equilibrium aerodynamic functions due to the elastic effects are $\bar{C}_x^{re}(\alpha)$, $\bar{C}_z^{re}(\alpha)$ and $\bar{C}_\eta^{re}(\alpha)$ defined by

$$\begin{pmatrix} \bar{C}_x^{re}(\alpha) \\ \bar{C}_z^{re}(\alpha) \\ \bar{C}_\eta^{re}(\alpha) \end{pmatrix} = \begin{pmatrix} -C_{x,\delta 1}\dfrac{C_{M,\eta}}{C_{M,\delta 1}}(\alpha) \\ C_{z,\eta} - C_{z,\delta 1}\dfrac{C_{M,\eta}}{C_{M,\delta 1}}(\alpha) \\ \bar{q}C_{\eta\eta} - C_{\eta\delta 1}\dfrac{C_{M,\eta}(\alpha)}{C_{M,\delta 1}(\alpha)} \end{pmatrix}.$$

Hence the equilibrium aerodynamic force vector in the u-w-η space at any equilibrium state is

$$\bar{F} = \begin{pmatrix} \bar{F}_x(u,w) + \bar{F}_x^e(u,w,\eta) \\ \bar{F}_z(u,w) + \bar{F}_z^e(u,w,\eta) \\ \bar{F}_\eta(u,w) + \bar{F}_\eta^e(u,w,\eta) \end{pmatrix} = \frac{1}{2}\rho V^2 S \begin{pmatrix} \bar{C}_x(\alpha) + \bar{C}_x^{re}(\alpha)\eta \\ \bar{C}_z(\alpha) + \bar{C}_z^{re}(\alpha)\eta \\ \bar{C}_\eta(\alpha, V) + \bar{C}_\eta^{re}(\alpha)\eta \end{pmatrix}.$$

Under practical circumstances an aircraft has a given flight envelope outside of which stability cannot be guaranteed or expected. This suggests dividing the analysis into two cases.

1. Assume unlimited control authority and work globally, or

2. Restrict analysis to a subset of states with adequate control authority.

Due to space limitations, in this paper we proceed with analysis for the global case only. The restriction on control authority restricts the results to a subset that is discussed elsewhere.

The system represented by (1) has three degrees of freedom and its three-dimensional equilibrium set M is characterized by solutions to the following equations

$$\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}-\mathrm{mg}\sin\theta_o+T_o\\ \mathrm{mg}\cos\theta_o\\ 0\end{pmatrix}+\frac{1}{2}\rho SV^2\begin{pmatrix}\bar{C}_x(\alpha)+\bar{C}_x^{re}(\alpha)\eta\\ \bar{C}_z(\alpha)+\bar{C}_z^{re}(\alpha)\eta\\ \bar{C}_\eta(\alpha,V)+\bar{C}_\eta^{re}(\alpha)\eta\end{pmatrix}+\begin{pmatrix}0\\0\\-\omega^2\eta\end{pmatrix} \tag{6}$$

where T, θ, and GW^* are parameters and m, S, ρ, ω and g are fixed constants.

Remark: For a given triplet (T,θ,GW^*), the equilibrium of system (1) is formed by all points

$$(u,w,q,\theta,\eta,\dot{\eta})=(u,w,0,\theta,\eta,0)$$

where

$$u=V\cos(\alpha),\quad w=V\sin(\alpha),$$

and (V,α,η) is a solution of (6) corresponding to the given (T,θ,GW^*). Thus given a triplet of (T,θ,GW^*), if (6) has a unique solution, then the system represented by (1) has a unique equilibrium point.

Dynamic Inversion

The control problem can be stated as follows

Problem statement: Given an equilibrium state $\bar{x}$, determine a controller $u=K(x,\bar{x})$ so that $\bar{x}$ is a global attractor for the system

$$\dot{x}=f\left(x,K(x,\bar{x})\right) \tag{7}$$

Any global attractor must be an equilibrium state. Using dynamic inversion this problem is addressed for vehicle models having a unique equilibrium point for appropriately chosen engine thrust T and mass distribution GW^*.

The approach is to invert the rotational dynamics to a stable set of desired dynamics. Since the throttle is typically a low-bandwidth control that is not changed during dynamic maneuvers, it is left fixed throughout the construction of the controller K and the analysis.

The philosophy behind the control law is for the vehicle to follow the pitch rate commands from the pilot and for the purpose of our analysis ignore the direct control of elastic modes. The structure of the dynamic inversion controller K is given by the following expression

$$\dot{q}^{des}=K1(q_{cmd}-q)+K2(\theta_{cmd}-\theta) \tag{8}$$

The desired dynamics are realized if the control surfaces conform to the following expression in the closed loop

$$\delta_1=\frac{1}{C_{M,\delta 1}(\alpha)}\left(\dot{q}^{des}\frac{2I_y}{\rho SV^2\bar{c}}-\left(C_M(\alpha)+C_{M,\eta}(\alpha)\eta+C_{M,\dot{\eta}}(\alpha)\dot{\eta}\right)\right) \tag{9}$$

Substituting the expression for controls (9) into the system equations (1) results in a closed loop system that readily separates into the following two subsystems. The controlled variable q and its associated equation gives

$$\begin{pmatrix}\dot q\\ \dot\theta\end{pmatrix}=\begin{pmatrix}0\\ q\end{pmatrix}+\frac{1}{2}\rho SV^2\begin{pmatrix}\bar c C_M(\cdot)/I_y\\ 0\end{pmatrix}+\frac{1}{2}\rho SV^2\begin{pmatrix}\bar c C_{M,\eta}(\cdot)/I_y\\ 0\end{pmatrix}\eta+\frac{1}{2}\rho SV^2\begin{pmatrix}\bar c C_{M,\dot\eta}(\cdot)/I_y\\ 0\end{pmatrix}\dot\eta$$
$$-\frac{1}{2}\rho SV^2\frac{\bar c}{I_y}\begin{pmatrix}C_M(\cdot)+C_{M,\eta}(\cdot)\eta+C_{M,\dot\eta}(\cdot)\dot\eta\\ 0\end{pmatrix}+\frac{1}{2}\rho SV^2\frac{\bar c}{I_y}\begin{pmatrix}\dfrac{2I_y}{\rho SV^2\bar c}\dot q^{des}\\ 0\end{pmatrix} \tag{10}$$
$$\rightarrow\begin{pmatrix}\dot q\\ \dot\theta\end{pmatrix}=\begin{pmatrix}0\\ q\end{pmatrix}+\begin{pmatrix}\dot q^{des}\\ 0\end{pmatrix}$$

These equations are linear in nature and decoupled from the u-w-η dynamics. The remaining dynamics of the u-w-η subsystem result in

$$\begin{pmatrix}\dot u\\ \dot w\\ \ddot\eta\end{pmatrix}=\begin{pmatrix}-qw\\ qu\\ 0\end{pmatrix}+\frac{1}{m_\eta}\begin{pmatrix}0\\ 0\\ -2\zeta\omega\dot\eta-\omega^2\eta\end{pmatrix}+\begin{pmatrix}-g\sin\theta+T/\mathrm{m}\\ g\cos\theta\\ 0\end{pmatrix}+\frac{1}{2}\rho SV^2\begin{pmatrix}\dfrac{\bar C_x(\alpha)}{\mathrm{m}}\\ \dfrac{\bar C_z(\alpha)}{\mathrm{m}}\\ \dfrac{C_\eta(\alpha,V)}{m_\eta}\end{pmatrix}+\frac{1}{2}\rho SV^2\begin{pmatrix}\dfrac{\bar C_x^{re}(\alpha)}{\mathrm{m}}\\ \dfrac{\bar C_z^{re}(\alpha)}{\mathrm{m}}\\ \dfrac{\bar C_\eta^{re}(\alpha)}{m_\eta}\end{pmatrix}\eta$$
$$+\frac{1}{2}\rho SV^2\begin{pmatrix}-\dfrac{C_{x,\delta 1}(\cdot)}{\mathrm{m}}\dfrac{C_{M,\eta}(\cdot)}{C_{M,\delta 1}(\cdot)}\\ \dfrac{C_{z,\eta}(\cdot)}{\mathrm{m}}-\dfrac{C_{z,\delta 1}(\cdot)}{\mathrm{m}}\dfrac{C_{M,\eta}(\cdot)}{C_{M,\delta 1}(\cdot)}\\ \dfrac{\bar q}{m_\eta}C_{\eta\eta}(\cdot)-\dfrac{C_{\eta\delta 1}(\cdot)}{m_\eta}\dfrac{C_{M,\eta}(\cdot)}{C_{M,\delta 1}(\cdot)}\end{pmatrix}\dot\eta+\frac{1}{2m_\eta}\rho SV^2\begin{pmatrix}0\\ 0\\ C_{\eta q}(\cdot)q\end{pmatrix}+\frac{1}{2}\rho SV^2\frac{1}{C_{M,\delta 1}(\cdot)}\begin{pmatrix}C_{x,\delta 1}(\cdot)/\mathrm{M}\\ C_{z,\delta 1}(\cdot)/\mathrm{M}\\ C_{\eta,\delta 1}(\cdot)/m_\eta\end{pmatrix}\frac{2I_y}{\rho SV^2\bar c}\dot q^{des} \tag{11}$$

where the equilibrium aerodynamic functions have been substituted to simplify the expression.
So proving the stability of the u-w-η subsystem would prove the stability of the entire closed loop system (10)-(11), since the commanded variable is stable by design.

Stability Result

We begin by showing the stability of the internal dynamics system (11), followed by establishing that the only possible orbits of the residualized system, derived later in the section , are equilibria. Then we establish that the residualized system has a global attractor, and finally put all of these together to show stability of the closed loop system dynamics (10)-(11).

Lemma 1: Assume the total aerodynamic drag, $\bar C_D(\alpha)=C_D(\alpha)-C_{D,\delta}(\alpha)\dfrac{C_M}{C_{M,\delta}}(\alpha)$, is always positive. Then there exists a finite neighborhood D centered at the origin of the u-w-η surface into which all trajectories enter and remain. And the dynamic system described by (11) is asymptotically stable.

Proof: Consider Lyapunov stability and the function $V^2+\dot\eta^2$ as a candidate Lyapunov function. If the candidate function proves to be a Lyapunov function then we can make conclusions about the asymptotic stability of the internal dynamics given by (11). Since the function is a quadratic, it is easy to verify that it will satisfy the first criteria of a Lyapunov function.

To satisfy the second criteria, from (11) we can compute the time rate of change of $V^2+\dot\eta^2=u^2+w^2+\dot\eta^2$:

$$\frac{d\left(V^2+\dot{\eta}^2\right)}{dt}=\frac{d\left(u^2+w^2+\dot{\eta}^2\right)}{dt}=2u\dot{u}+2w\dot{w}+2\dot{\eta}\ddot{\eta}$$

$$=2\dot{\eta}\left(-2\zeta\omega\dot{\eta}-\omega^2\eta+\frac{C_{\eta,\delta 1}(\cdot)}{C_{M,\delta 1}(\cdot)}\frac{I_y}{m_\eta \bar{c}}\dot{q}^{des}\right)+2V\left(g\sin(\theta-\alpha)-\frac{T}{\mathrm{m}}\cos\alpha-\frac{C_{D,\delta 1}(\cdot)}{C_{M,\delta 1}(\cdot)}\frac{I_y}{\bar{c}\mathrm{m}}\dot{q}^{des}\right)$$

$$+\frac{\rho S V^2}{m_\eta}\left(\bar{C}_\eta^{re}(\alpha)\eta\dot{\eta}+\left(\bar{C}_\eta(\alpha,V)+C_{\eta q}(\cdot)q\right)\dot{\eta}+\left(\bar{q}C_{\eta\eta}(\cdot)-C_{\eta\delta 1}(\cdot)\frac{C_{M,\eta}(\cdot)}{C_{M,\delta 1}(\cdot)}\right)\dot{\eta}^2\right)$$

$$+\frac{\rho S V^3}{\mathrm{m}}\left(-\bar{C}_D(\alpha)-\bar{C}_D^{re}(\alpha)\eta+\left(C_{D,\delta 1}(\cdot)\frac{C_{M,\eta}(\cdot)}{C_{M,\delta 1}(\cdot)}-\left(\sin\alpha\cos\alpha C_{L,\eta}(\alpha)+\sin^2\alpha C_{D,\eta}(\alpha)\right)\right)\dot{\eta}\right)$$

where $\dot{q}^{des}$ is defined in (8) and

$$\bar{C}_D=-\sin\alpha\bar{C}_z-\cos\alpha\bar{C}_x \qquad C_{D,\delta 1}=-\sin\alpha C_{z,\delta 1}-\cos\alpha C_{x,\delta 1} \qquad \bar{C}_D^{re}=-\sin\alpha\bar{C}_z^{re}-\cos\alpha\bar{C}_x^{re}$$

$$\bar{C}_L=-\cos\alpha\bar{C}_z+\sin\alpha\bar{C}_x \qquad C_{L,\eta}=-\cos\alpha C_{z,\eta}+\sin\alpha C_{x,\eta} \qquad C_{D,\eta}=-\sin\alpha C_{z,\eta}-\cos\alpha C_{x,\eta}$$

The expression for $\bar{C}_D$ denotes the total drag coefficient including direct surface effects. For sufficiently large V, the V^3 term will dominate the candidate Lyapunov function. Hence, if the coefficient of V^3 is negative, the time rate of change of $V^2+\dot{\eta}^2$ is negative and the conclusions of asymptotic stability of (11) follow.

In order for the coefficient of V^3 to be negative, with the assumption that $\bar{C}_D$ is positive, which is borne out in practice, the following expression must hold true:

$$\bar{C}_D(\alpha)>\left|-\bar{C}_D^{re}(\alpha)\eta+\left(C_{D,\delta 1}(\cdot)\frac{C_{M,\eta}(\cdot)}{C_{M,\delta 1}(\cdot)}-\left(\sin\alpha\cos\alpha C_{L,\eta}(\alpha)+\sin^2\alpha C_{D,\eta}(\alpha)\right)\right)\dot{\eta}\right|$$

This places limitations on the size of the elastic deformation and velocity.

As has been observed earlier, the dynamics of q and θ are decoupled from the dynamics of velocity $(u,w,\dot{\eta})$. Therefore, while studying the internal dynamics, it may be assumed that q and θ are known functions and hence (11) becomes a three dimensional time-varying nonlinear system.

Furthermore by design and physical limitation of the aircraft $q\to 0$ and $\theta\to\theta_{cmd}$ as $t\to\infty$, so the internal dynamics in (11) become

$$\begin{pmatrix}\dot{u}\\ \dot{w}\\ \ddot{\eta}\end{pmatrix}=\begin{pmatrix}-g\sin\theta_{cmd}+T/\mathrm{m}\\ g\cos\theta_{cmd}\\ -\omega^2\eta-2\zeta\omega\dot{\eta}\end{pmatrix}+\begin{pmatrix}\left(\bar{F}_x(u,w)+\bar{F}_x^e(u,w,\eta)\right)/\mathrm{m}\\ \left(\bar{F}_z(u,w)+\bar{F}_z^e(u,w,\eta)\right)/\mathrm{m}\\ \left(\bar{F}_\eta(u,w)+\bar{F}_\eta^e(u,w,\eta)\right)/m_\eta\end{pmatrix}+\frac{1}{2}\rho S V^2\begin{pmatrix}-\dfrac{C_{x,\delta 1}(\alpha)}{\mathrm{m}}\dfrac{C_{M,\eta}(\alpha)}{C_{M,\delta 1}(\alpha)}\\ \dfrac{C_{z,\eta}(\alpha)}{\mathrm{m}}-\dfrac{C_{z,\delta 1}(\alpha)}{\mathrm{m}}\dfrac{C_{M,\eta}(\alpha)}{C_{M,\delta 1}(\alpha)}\\ \dfrac{\bar{q}}{m_\eta}C_{\eta\eta}-\dfrac{C_{\eta\delta 1}(\alpha)}{m_\eta}\dfrac{C_{M,\eta}(\alpha)}{C_{M,\delta 1}(\alpha)}\end{pmatrix}\dot{\eta} \tag{12}$$

where $\bar{F}_i,\bar{F}_i^e$ for $i=x,z,\eta$ have been substituted from (6). Mathematically, the residualized system (12) is called the limiting system of system (11) whose dynamics behavior determines that of (11)[5] and is discussed in detail in reference 6.

Lemma 2: Define G as the right hand side of (12), $\bar{F}=\left(\bar{F}_i+\bar{F}_i^{re}+f_i^{\eta}\right)$ and $\tilde{M}=diag(\mathrm{m}^{-1},\mathrm{m}^{-1},m_\eta^{-1})$. Assume that the aerodynamic forces have a dissipative effect, that is

$$div(G) < 0$$

or equivalently (13)

$$\omega^2 + \frac{\rho SV}{2m}\left(3\bar{C}_D + \frac{d\bar{C}_L}{d\alpha}\right)(\alpha) - \frac{\rho V^2 S}{2m_\eta}\bar{C}_\eta^{re}(\alpha,V) > \left|\frac{\rho SV}{2m}\left(\left(3\bar{C}_D^{re} + \frac{d\bar{C}_L^{re}}{d\alpha}\right)(\alpha)\eta + \left((1+\sin^2\alpha)\bar{C}_{D,\eta} + \frac{d\bar{C}_{L,\eta}}{d\alpha}\right)(\alpha)\dot{\eta}\right)\right|$$

for all (u, w, η) not equal to (0,0,0). Then the only possible orbits of the residualized system are equilibria.

Proof:
The proof is by contradiction and application of the Divergence Theorem. Assume that there exists a closed orbit C and let R define the interior region, then by the Divergence Theorem

$$\iint_C G \cdot nd\sigma = \iiint_R divG\, d\, Volume$$

where $\boldsymbol{n}d\sigma$ is a vector element of surface area directed along the unit outer normal vector $\boldsymbol{n}$ to C. The inner product on the left side is zero by construction, while $divG$ on the right is equal to $-\omega^2 + div(\tilde{M}\bar{F})$ and is negative by assumption. Hence, there exists a contradiction which proves the lemma. (This Lemma is an extension of the theorem in Hale (1980) chapter 2, Exercese 1.3.)

As a corollary of the above dissipative condition lemma, the following stability result applies to the residualized system (12).

Corollary: Assume that

1. the residualized system (12) has a unique equilibrium (u_o, w_o, η_o) for given T, θ_{cmd}, and GW^*;
2. the residualized aerodynamic force vector $\bar{F}$ satisfies the divergent condition (13).

Then the unique equilibrium is a global attractor for the residualized system (12), i.e. any solution $(u(t), w(t), \eta(t))$ of the residualized system (12) satisfies.

$$\lim_{t\to\infty} u(t) = u_o, \quad \lim_{t\to\infty} w(t) = w_o, \quad \lim_{t\to\infty} \eta(t) = \eta_o$$

Proof: By Lyapunov stability result of Lemma 1 any solution $(u(t), w(t), \eta(t))$ is bounded. Thus, its limit set is either an equilibrium or a periodic orbit. But Lemma 2 excludes the periodic orbit case. Hence, the limit set of any solution consists of one point (u_o, w_o, η_o) which is the unique equilibrium of (12) by the assumption. This is equivalent of saying that (u_o, w_o, η_o) is the global attractor of the system (12).

Remark: For a given T, θ_{cmd}, and GW^*; the residualized system (12) has a unique equilibrium if and only if the equations

$$\begin{aligned}
0 &= (-g\sin\theta_{cmd} + T/\mathrm{m})\omega^2 - \frac{1}{2m_\eta}\rho SV^2\bar{C}_\eta^{re}(\alpha)(-g\sin\theta_{cmd} + T/\mathrm{M}) + \frac{1}{2\mathrm{m}}\rho SV^2\bar{C}_x(\alpha)\omega^2 \\
&\quad + \frac{1}{4\mathrm{m}m_\eta}\rho^2 S^2 V^4\left(\bar{C}_x^{re}(\alpha)\bar{C}_\eta(\alpha,V) - \bar{C}_x(\alpha)\bar{C}_\eta^{re}(\alpha)\right) \\
0 &= g\cos\theta_{cmd}\omega^2 - \frac{1}{2m_\eta}\rho SV^2\bar{C}_\eta^{re}(\alpha)g\cos\theta_{cmd} + \frac{1}{2\mathrm{m}}\rho SV^2\bar{C}_z(\alpha)\omega^2 \\
&\quad + \frac{1}{4\mathrm{m}m_\eta}\rho^2 S^2 V^4\left(\bar{C}_z^{re}(\alpha)\bar{C}_\eta(\alpha,V) - \bar{C}_z(\alpha)\bar{C}_\eta^{re}(\alpha)\right)
\end{aligned} \tag{14}$$

have only one solution (α_o, V_o) for given T, θ_{cmd}, and GW^* results from residualizing (12) further to remove η. Solution can be found by looking for an intersection of contour plots of each equation.

The culmination of lemmas and corollary combine into the main stability result. The stability result for the closed loop system by (10)-(11) is given in the following theorem:

Theorem: Assume that

1. the total drag coefficient

$$\bar{C}_D(\alpha) = C_D(\alpha) - C_{D,\delta}(\alpha)\frac{C_M}{C_{M,\delta}}(\alpha) > 0$$

and

$$\bar{C}_D(\alpha) > \left| -\bar{C}_D^{re}(\alpha)\eta + \left(C_{D,\delta 1}(\cdot)\frac{C_{M,\eta}(\cdot)}{C_{M,\delta 1}(\cdot)} - \left(\sin\alpha\cos\alpha C_{L,\eta}(\alpha) + \sin^2\alpha C_{D,\eta}(\alpha)\right)\right)\dot{\eta} \right|,$$

2. the aerodynamic functions satisfy the dissipative condition

$$div(G) < 0$$

$$-\omega^2 - \frac{\rho SV}{2\text{m}}\left(3\bar{C}_D(\alpha) + \frac{d\bar{C}_L(\alpha)}{d\alpha} + \left(3\bar{C}_D^{re}(\alpha) + \frac{d\bar{C}_L^{re}(\alpha)}{d\alpha} \right)\eta + \left((1+\sin^2\alpha)\bar{C}_{D,\eta}(\alpha) + \frac{d\bar{C}_{L,\eta}(\alpha)}{d\alpha} \right)\dot{\eta} \right)$$

$$+\frac{\rho V^2 S}{2m_\eta}\bar{C}_\eta^{re}(\alpha, V) < 0$$

or

$$\omega^2 + \frac{\rho SV}{2\text{m}}\left(3\bar{C}_D + \frac{d\bar{C}_L}{d\alpha} \right)(\alpha) - \frac{\rho V^2 S}{2m_\eta}\bar{C}_\eta^{re}(\alpha, V) > \left| \frac{\rho SV}{2\text{m}}\left(\left(3\bar{C}_D^{re} + \frac{d\bar{C}_L^{re}}{d\alpha} \right)(\alpha)\eta + \left((1+\sin^2\alpha)\bar{C}_{D,\eta} + \frac{d\bar{C}_{L,\eta}}{d\alpha} \right)(\alpha)\dot{\eta} \right) \right|$$

with G defined as the right side of residualized equation (12),

3. for given T, θ_{cmd}, and GW^*, the following equations

$$0 = \left(-g\sin\theta_{cmd} + T/\text{m}\right)\omega^2 - \frac{1}{2m_\eta}\rho SV^2\bar{C}_\eta^{re}(\alpha)\left(-g\sin\theta_{cmd} + T/\text{M}\right) + \frac{1}{2\text{m}}\rho SV^2\bar{C}_x(\alpha)\omega^2$$

$$+\frac{1}{4\text{m}m_\eta}\rho^2 S^2 V^4\left(\bar{C}_x^{re}(\alpha)\bar{C}_\eta(\alpha, V) - \bar{C}_x(\alpha)\bar{C}_\eta^{re}(\alpha)\right)$$

$$0 = g\cos\theta_{cmd}\omega^2 - \frac{1}{2m_\eta}\rho SV^2\bar{C}_\eta^{re}(\alpha)g\cos\theta_{cmd} + \frac{1}{2\text{m}}\rho SV^2\bar{C}_z(\alpha)\omega^2$$

$$+\frac{1}{4\text{m}m_\eta}\rho^2 S^2 V^4\left(\bar{C}_z^{re}(\alpha)\bar{C}_\eta(\alpha, V) - \bar{C}_z(\alpha)\bar{C}_\eta^{re}(\alpha)\right)$$

have only one solution (α_o, V_o).

Then, for a given T, θ_{cmd}, and GW^*, the closed loop system (10)-(11) has a unique equilibrium $(u_o, w_o, q_o, \theta_o, \eta_o, \dot{\eta}_o)$ given by

$$q_o = 0, \qquad \theta_o = \theta_{cmd}$$

and

$$u_o = \cos\alpha_o\left[\frac{\left(\omega^2\bar{C}_z - g\cos\theta_{cmd}\bar{C}_\eta^{re}\right) + \sqrt{\left(\omega^2\bar{C}_z - g\cos\theta_{cmd}\bar{C}_\eta^{re}\right)^2 - 4\omega^2 g\cos\theta_{cmd}\left(\bar{C}_z\bar{C}_\eta^{re} - \bar{C}_z^{re}\bar{C}_\eta\right)}}{\left(\bar{C}_z\bar{C}_\eta^{re} - \bar{C}_z^{re}\bar{C}_\eta\right)}\right]^{1/2}(\alpha_o)$$

$$w_o = \sin\alpha_o \left[\frac{\left(\omega^2 \bar{C}_z - g\cos\theta_{cmd}\bar{C}_\eta^{re}\right) + \sqrt{\left(\omega^2\bar{C}_z - g\cos\theta_{cmd}\bar{C}_\eta^{re}\right)^2 - 4\omega^2 g\cos\theta_{cmd}\left(\bar{C}_z\bar{C}_\eta^{re} - \bar{C}_z^{re}\bar{C}_\eta\right)}}{\left(\bar{C}_z\bar{C}_\eta^{re} - \bar{C}_z^{re}\bar{C}_\eta\right)} \right]^{1/2} \quad (\alpha_o)$$

$$\eta_o = \frac{1}{2}\rho S V_o^2 \frac{\bar{C}_\eta(\alpha_o, V_o)}{2\omega^2/\rho S V_o^2 - \bar{C}_\eta^{re}(\alpha_o)} \qquad \dot{\eta}_o = 0$$

where α_o is from a unique solution of the equations (14).
Furthermore, any solution $\left(u(t), w(t), q(t), \theta(t), \eta(t), \dot{\eta}(t)\right)$ of the closed loop system (10)-(11) satisfies

$u(t) \to u_o,\quad w(t) \to w_o,\quad q(t) \to 0,\quad \theta(t) \to \theta_{cmd},\quad \eta(t) \to \eta_o,\quad \dot{\eta}(t) \to 0$

as $t \to \infty$. In other words $\left(u_o, w_o, 0, \theta_{cmd}, \eta_o, 0\right)$ is a global attractor of the closed loop system (10)-(11).

Proof: A basic sketch of the proof is given here. A more rigorous and detailed version is presented elsewhere due to space limitation [6].
By the design of the feedback control the closed loop system is decoupled into internal dynamics (11) and controlled dynamics (10), which are stable. Furthermore, since $q(t) \to 0$ and $\theta(t) \to \theta_{cmd}$ as $t \to \infty$, internal dynamics system (11) is asymptotically autonomous and its limiting equation is given by the residualized system (12). We invoke the Markus theorem[7] to relate the stability of (11) to that of (12) which establishes that $\left(u_o, w_o, \eta_o\right)$ is the local attractor to the time varying system (11). Then we apply the Yoshizawa theorem[8] to show that it is a global attractor of (11).

The first two assumptions are based on the physical characteristics of the vehicle under consideration. Furthermore, a variation of these assumptions that does not include the flexible vehicle dynamics, have been shown by Morton, et. al. to hold for a variety of fighter aircraft. Hence, the stability of the closed loop system depends on whether equations (14) have a unique solution (α_o, V_o). This in turn depends on the uniqueness of α for a given engine thrust T and the commanded attitude θ.

In applying the stability results to the HSCT aircraft we take into consideration the structure of the aircraft model. The dynamic aeroelastic data is derived in a linear fashion and is then combined with nonlinear rigid body aerodynamic model[9]. The implications of this nonlinear/linear conglomeration on system shown in (1) is the removal of functional dependence of all η and $\dot{\eta}$ terms on V. They all become linear functions of α. Define $E_{(\cdot)}$ as a linear representation for the aerodynamic coefficients $C_{(\cdot)}$, then the following replacements are made in system (1):

$$Z_\eta = \frac{1}{2}\rho S V^2 C_{z,\eta}(\alpha, \mathrm{M}, GW^*) \qquad Z_\eta = \frac{1}{2}\rho S V^2 C_{z,\eta}(\alpha, \mathrm{M}, GW^*)$$

$$M_\eta = \frac{1}{2}\rho S V^2 \bar{c} C_{M,\eta}(\alpha, \mathrm{M}, GW^*) \qquad M_\eta = \frac{1}{2}\rho S V^2 \bar{c} C_{M,\eta}(\alpha, \mathrm{M}, GW^*)$$

$$E_\eta = \frac{1}{2}\rho S V^2 \bar{q} C_{\eta\eta}(\mathrm{M}, GW^*) \qquad E_\eta = \frac{1}{2}\rho S V^2 \bar{q} C_{\eta\eta}(\mathrm{M}, GW^*)$$

$$E_u u + E_w w + E_q q = \frac{1}{2}\rho S V^2 \left(C_{\eta u}(\mathrm{M}, GW^*)u + C_{\eta w}(\mathrm{M}, GW^*)w + C_{\eta q}(\mathrm{M}, GW^*)q\right)$$

$$E_{\delta 1} = \frac{1}{2}\rho S V^2 C_{\eta\delta 1}(\mathrm{M}, GW^*)$$

The aerodynamic functions for both nonlinear rigid body dynamics and dynamic aeroelastic contribution are shown in figures 2 and 3. Taking advantage of the model structure, assumption (1) of the stability theorem reduces to

$$\bar{C}_D(\alpha) = C_D(\alpha) - C_{D,\delta}(\alpha)\frac{C_M}{C_{M,\delta}}(\alpha) > 0$$

and assumption (2) becomes

$$\omega^2 + \frac{\rho VS}{2\mathrm{m}}\left(3\bar{C}_D(\alpha) + \frac{d\bar{C}_L(\alpha)}{d\alpha}\right) > \left|\left(-\bar{E}_D^{re} - \frac{d\bar{E}_L^{re}}{d\alpha}\right)(\alpha)\frac{\eta}{\mathrm{m}V} + \frac{M_\eta}{\mathrm{m}\bar{c}}\frac{\dot{\eta}}{V}\left(\frac{C_{D,\delta 1}}{C_{M,\delta 1}}(\alpha) + \frac{d\left(\frac{C_{L,\delta 1}}{C_{M,\delta 1}}(\alpha)\right)}{d\alpha}\right) + \frac{1}{m_\eta}\bar{E}_\eta^{re}(\alpha)\right|$$

Figure 4 illustrates how the assumptions are satisfied for a sample flight condition chosen to produce the worse case dynamic aeroelastic interactions. In assumption 2, the coefficients of the all η and $\dot{\eta}$ terms are several orders of magnitude smaller than $\bar{E}_\eta^{re}(\alpha)$. By model construction $m_\eta = 1$, thus making $\bar{E}_\eta^{re}(\alpha)$ a completely dominant term on the right hand side of the inequality. From a strictly mathematical perspective, one might argue that η and especially $\dot{\eta}$ can become large enough to rival the dominance of $\bar{E}_\eta^{re}(\alpha)$. If that were the case, then in the physical world the vehicle would have suffered a critical structural failure. For illustrative purposes, we assume $\dot{\eta} = 1000\eta = \sim 120 ft/\sec$ and show in figure 4 the difference between left and right hand sides of the inequality.

Conclusions

In this paper, we started to establish conditions under which the dynamic inversion control methodology can be guaranteed global stability when applied to a real world problem. The problem under consideration is an advanced high-speed, large flexible aircraft that requires the issue of flexibility to be to addressed as part of flight control design. The initial approach is to simplify the problem, address the pitch axis dynamics and control only one flight variable. The role played by flexible dynamics is immediately apparent from consideration of the internal dynamics of the system. Furthermore, these flexible dynamics play a role in establishing stability guarantees for the closed loop system.

The results presented are the first to include flexible dynamics in stability analysis of dynamic inversion methodology. These form an initial basis to more complicated control problem formulation that includes an integrated structural/flight control that is essential for this class of vehicles. The results of this work will be presented in subsequent publication.

Figures

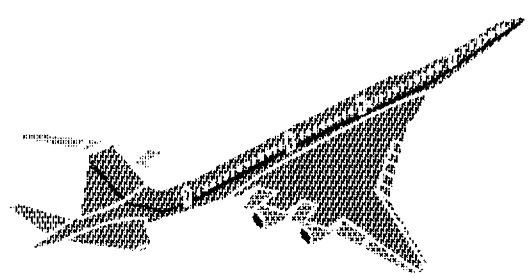

Figure 1. HSCT candidate vehicle configuration.

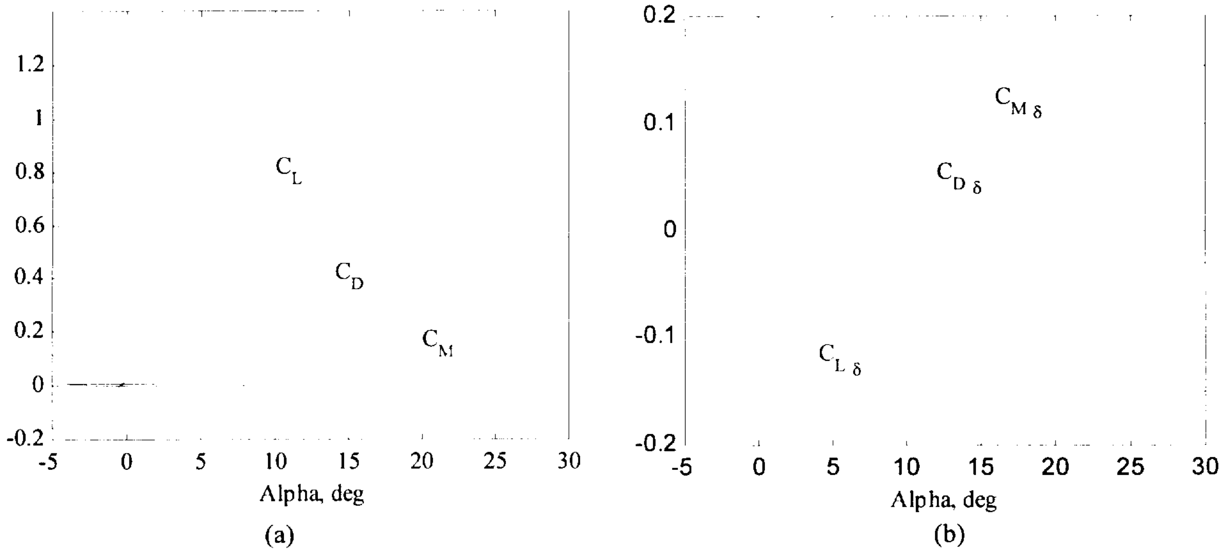

Figure 2. Nondimensional Aerodynamic Functions (a) Lift, Drag and Pitching Moment Coefficients and (b) Lift, Drag, and Pitching Moment Control Coefficients

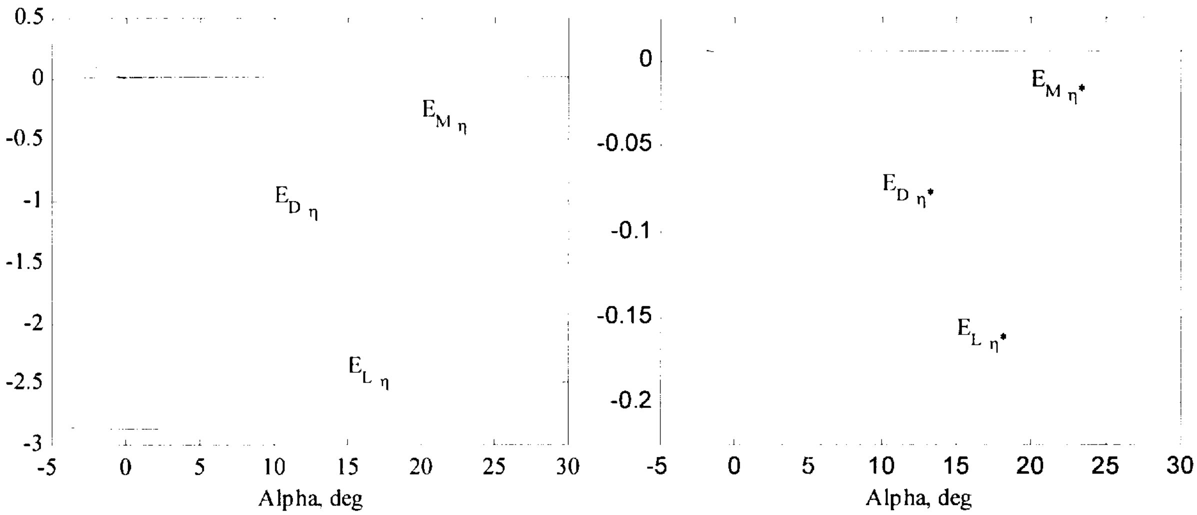

Figure 3. Dimensional dynamic aeroelastic contributions to Lift, Drag, and Pitching Moment due to elastic deformation, η, and rate of change of the deformation, $d\alpha/dt = \eta^*$.

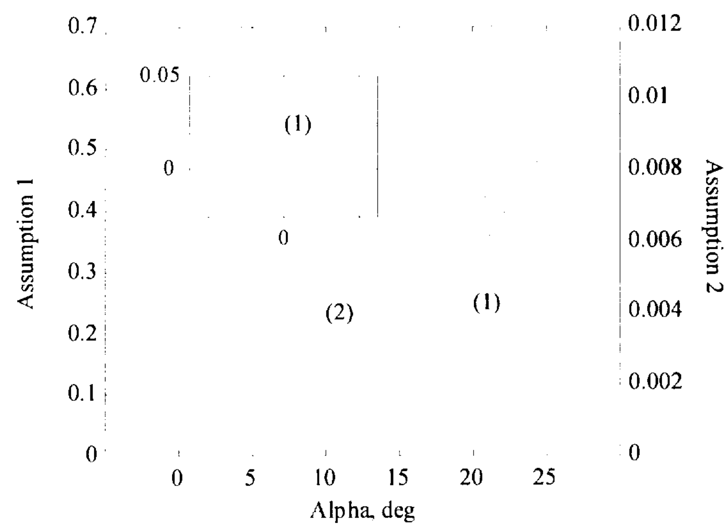

Figure 4. Assumptions 1 and 2

(1) Assumption 1: $\bar{C}_D = C_D - C_{D,\delta 1}\frac{C_M}{C_{M,\delta 1}} > 0$

(2) Assumption 2: $div(G) < 0$

$$\omega^2 + \frac{\rho VS}{2m}\left(3\bar{C}_D(\alpha)+\frac{d\bar{C}_L(\alpha)}{d\alpha}\right) - \left|\left(-\bar{E}_D^{re} - \frac{d\bar{E}_L^{re}}{d\alpha}\right)(\alpha)\frac{\eta}{mV} + \frac{M_\eta}{m\bar{c}}\frac{\dot{\eta}}{V}\left(\frac{C_{D,\delta 1}}{C_{M,\delta 1}}(\alpha) + \frac{d\left(\frac{C_{L,\delta 1}}{C_{M,\delta 1}}(\alpha)\right)}{d\alpha}\right) + \frac{1}{m_\eta}\bar{E}_\eta^{re}(\alpha)\right| > 0$$

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