NASA TECHNICAL NOTE

NASA TN D-2894

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • JULY 1965

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SUMMARY

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State-vector control has been applied to the problem of lateral stability augmentation of high-performance aircraft. The problem considered was that of evaluating the feedback gains that would produce desired stability characteristics. The feedback gains are obtained as solutions of linear algebraic equations. A numerical example illustrates the practicability of the method.

Author

INTRODUCTION

Perturbed lateral motions of a high-performance aircraft may have undesirable stability characteristics at some flight conditions. A linear feedback control system usually is designed to augment the lateral stability. The design problem is quite complex if deflections of either the ailerons or the rudder produce changes in both the rolling and yawing moments. The cross-control effects (yawing moment caused by aileron deflection and rolling moment caused by rudder deflection), if large, may require that the feedback of the rolling and yawing velocities be applied to the ailerons and rudder to provide additional damping. The designer must determine the required cross-damper feedback gains (rudder deflection due to rolling velocity and aileron deflection due to yawing velocity) that provide desired stability characteristics, in addition to the usual yaw-damper and roll-damper feedbacks.

A well-known mathematical theorem on linear dynamical systems (ref. 1) will be applied to derive analytical expressions for the gains that will produce any desired roots of the characteristic equation of the augmented system. The feedback gains are obtained as solutions of linear algebraic equations if the linear dynamical system has the property of being completely controllable using a single linear control function. The property of complete controllability has a precise mathematical definition, and it is possible to check this property of the system and to know in advance whether or not the required gains can be computed. A numerical example is included for illustration.

SYMBOLS

b	span, ft
C_l	rolling-moment coefficient, $\frac{\text{Rolling moment}}{qSb}$
C_n	yawing-moment coefficient, $\frac{\text{Yawing moment}}{qSb}$
C_Y	side-force coefficient, $\frac{\text{Side force}}{qS}$
$\vec{c}$	constant vector with components c_1 and c_2
g	acceleration due to gravity, 32.2 ft/sec^2
I_X	moment of inertia about principal body X-axis, slug-ft^2
I_Z	moment of inertia about principal body Z-axis, slug-ft^2
$j = \sqrt{-1}$	
$\vec{k}$	feedback gain vector with components k_i where i is 1, 2, 3, and 4
m	mass, slugs
q	dynamic pressure, $\frac{1}{2}\rho V^2$, lb/sq ft
S	wing area, sq ft
t	time, sec
$\vec{u}$	control vector with components δ_a and δ_r
V	airspeed, ft/sec
$\vec{x}$	state vector with components x_1, x_2, x_3, and x_4
α	angle of attack of principal X-axis
β	angle of sideslip
δ_a	aileron deflection
δ_r	rudder deflection

λ parameter

λ_i roots of characteristic equation where i is 1, 2, 3, and 4

ρ air density, slugs/cu ft

ϕ angle of roll

ψ angle of yaw

$$C_{l_\beta} = \frac{\partial C_l}{\partial \beta}$$

$$C_{l_{\delta_a}} = \frac{\partial C_l}{\partial \delta_a}$$

$$C_{l_{\delta_r}} = \frac{\partial C_l}{\partial \delta_r}$$

$$C_{n_\beta} = \frac{\partial C_n}{\partial \beta}$$

$$C_{n_{\delta_a}} = \frac{\partial C_n}{\partial \delta_a}$$

$$C_{n_{\delta_r}} = \frac{\partial C_n}{\partial \delta_r}$$

$$C_{Y_\beta} = \frac{\partial C_Y}{\partial \beta}$$

All angles are in radians unless otherwise noted. One dot over a quantity means the first derivative with respect to time, and two dots over a quantity means the second derivative with respect to time.

STATE-VECTOR EQUATIONS OF MOTION

A system of linear differential equations with constant coefficients that describe the perturbed lateral motion of an airplane (see ref. 2) may be written, using principal body axes, as

$$
\left.\begin{aligned}
&\dot{\beta} - \frac{qS}{mV} C_{Y_\beta}\beta - \frac{g}{V}\phi - \alpha\dot{\phi} + \dot{\psi} = 0 \\
&\ddot{\phi} - \frac{qSb}{I_X} C_{l_\beta}\beta - \frac{qSb}{I_X} C_{l_{\delta_a}}\delta_a - \frac{qSb}{I_X} C_{l_{\delta_r}}\delta_r = 0 \\
&\ddot{\psi} - \frac{qSb}{I_Z} C_{n_\beta}\beta - \frac{qSb}{I_Z} C_{n_{\delta_a}}\delta_a - \frac{qSb}{I_Z} C_{n_{\delta_r}}\delta_r = 0
\end{aligned}\right\} \tag{1}
$$

Terms involving the rotary derivatives and the effect of control authorities on side force have been omitted for convenience. These terms could have been included without affecting the method of analysis, except that computations would be more complicated.

If $x_1 = \beta$, $x_2 = \phi$, $x_3 = \dot{\phi}$, and $x_4 = \dot{\psi}$, then the state of the system may be expressed as a four-component vector $\vec{x}$, or

$$
\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \beta \\ \phi \\ \dot{\phi} \\ \dot{\psi} \end{bmatrix} \tag{2}
$$

Similarly, the control deflections may be written as a two-component vector $\vec{u}$, or

$$
\vec{u} = \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix} \tag{3}
$$

The system (eq. (1)), with equations (2) and (3), may be written in vector-matrix notation, as

$$
\dot{\vec{x}} = [A]\vec{x} + [G]\vec{u} \tag{4}
$$

where

$$[A] = \begin{bmatrix} \frac{qS}{mV} C_{Y_\beta} & \frac{g}{V} & \alpha & -1 \\ 0 & 0 & 1 & 0 \\ \frac{qSb}{I_X} C_{l_\beta} & 0 & 0 & 0 \\ \frac{qSb}{I_Z} C_{n_\beta} & 0 & 0 & 0 \end{bmatrix}$$

and

$$[G] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{qSb}{I_X} C_{l_{\delta_a}} & \frac{qSb}{I_X} C_{l_{\delta_r}} \\ \frac{qSb}{I_Z} C_{n_{\delta_a}} & \frac{qSb}{I_Z} C_{n_{\delta_r}} \end{bmatrix}$$

The elements of the matrices $[A]$ and $[G]$ are constant parameters that depend on flight conditions and airplane characteristics. The matrix $[G]$ is the control effectiveness matrix.

The system of equations (eq. (4)) is the state-vector form of the equations that describe the perturbed lateral motion of an airplane with controls. Augmentation of the stability of the airplane is analyzed in the following discussion.

STABILITY AUGMENTATION

Consider the airplane equations of motion with $\vec{u} = 0$. The system of equations of motion for the free system is

$$\dot{\vec{x}} = [A]\vec{x} \tag{5}$$

The characteristic equation for the free system is a polynomial equation in λ and is defined by

$$p(\lambda) = \det\left[\lambda[I] - [A]\right] = 0 \tag{6}$$

or

$$p(\lambda) = \lambda^4 + \frac{qS}{mV} C_{Y_\beta}\lambda^3 + qSb\left(\frac{1}{I_Z} C_{n_\beta} - \frac{\alpha}{I_X} C_{l_\beta}\right)\lambda^2 - \frac{g}{V}\frac{qSb}{I_X} C_{l_\beta}\lambda = 0 \tag{7}$$

where $[I]$ is the identity matrix, λ is a parameter, and $\det[\;]$ means determinant of a square matrix. The roots λ_i of the characteristic equation provide information about the stability of the airplane.

If the free airplane system has undesirable stability characteristics, then a feedback control system may be desirable to augment the stability. The problem, therefore, is to choose the control vector $\vec{u}$ such that the augmented system (eq. (4)) has the desired stability characteristics, or, equivalently, that the roots of the characteristic equation of the augmented system have the desired values.

A mathematically simple form of the control vector results if the components δ_a and δ_r are assumed to depend only on a single linear feedback function and to vary proportionally. The control vector then can be written

$$\vec{u} = \vec{c}(\vec{k} \cdot \vec{x}) = \vec{c}\,\vec{k}^*\,\vec{x} \tag{8a}$$

or

$$\left.\begin{aligned} \delta_a &= c_1\left(k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4\right) \\ \delta_r &= c_2\left(k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4\right) \end{aligned}\right\} \tag{8b}$$

In equation (8a), the linear feedback function $(\vec{k} \cdot \vec{x})$ is the dot product of the vectors $\vec{k}$ and $\vec{x}$, and $\vec{k}^*$ is the transpose of $\vec{k}$. The vector $\vec{k}$ is called the feedback gain vector and its components k_i are the feedback gains. The components of the constant vector $\vec{c}$ are c_1 and c_2. It is clear from equations (8b) that δ_r and δ_a will generally vary proportionately $\left(\text{when } c_1 \neq 0 \text{ and } c_2 \neq 0\right)$. However, for augmentation with either rudder or ailerons alone, either c_1 or c_2 may be identically zero with the other component equal to one. In any case, it is required that all four state-vector components (β, ϕ, $\dot{\phi}$, and $\dot{\psi}$) be fed back.

The system of equations of motion (eq. (4)), with the control vector defined in equation (8a), is

$$\dot{\vec{x}} = [A]\vec{x} + [G]\vec{c}\,\vec{k}^*\,\vec{x} = \left[[A] + [G]\vec{c}\,\vec{k}^*\right]\vec{x} \tag{9}$$

Note that $\left[[G]\vec{c}\,\vec{k}^*\right]$ is an $n \times n$ matrix and that $\left[[A] + [G]\vec{c}\,\vec{k}^*\right]$ is the matrix of the augmented system. The characteristic equation for the augmented system (eq. (9)) is also a polynomial equation in λ and is

$$q(\lambda) = \det\left[\lambda[I] - [A] - [G]\,\vec{c}\,\vec{k}^*\right] = 0 \tag{10}$$

Reference 1 shows that for each choice of characteristic roots, a gain vector $\vec{k}$ can be determined if a condition of complete controllability, as defined in reference 3, is satisfied. Furthermore, the gain vector is computed by solving a set of linear algebraic equations in the gain factors. A brief discussion of complete controllability of an augmented system is presented in the appendix.

The determinant of equation (10), when expanded, gives $q(\lambda)$ as

$$\begin{aligned} q(\lambda) = {} & \lambda^4 - \left(L_c k_3 + N_c k_4 + Y_\beta\right)\lambda^3 \\ & - \left[\alpha L_\beta - N_\beta + \left(\alpha L_c - N_c\right)k_1 + L_c k_2 - Y_\beta L_c k_3 - Y_\beta N_c k_4\right]\lambda^2 \\ & - \left[\frac{g}{V} L_\beta + \frac{g}{V} L_c k_1 - Y_\beta L_c k_2 + \left(N_\beta L_c - L_\beta N_c\right)k_3 + \alpha\left(N_\beta L_c - L_\beta N_c\right)k_4\right]\lambda \\ & - \left(N_\beta L_c - L_\beta N_c\right)k_2 - \frac{g}{V}\left(N_\beta L_c - L_\beta N_c\right)k_4 = 0 \end{aligned} \tag{11a}$$

where

$$\left.\begin{aligned} Y_\beta &= \frac{qS}{mV} C_{Y_\beta} \\ L_c &= \left(c_1 C_{l_{\delta_a}} + c_2 C_{l_{\delta_r}}\right)\frac{qSb}{I_X} \\ N_c &= \left(c_1 C_{n_{\delta_a}} + c_2 C_{n_{\delta_r}}\right)\frac{qSb}{I_Z} \\ L_\beta &= \frac{qSb}{I_X} C_{l_\beta} \\ N_\beta &= \frac{qSb}{I_Z} C_{n_\beta} \end{aligned}\right\} \tag{11b}$$

The polynomial $q(\lambda)$ also may be written in terms of the desired characteristic roots λ_i as

$$q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$$

$$= \lambda^4 - (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)\lambda^3 + \left[\lambda_1\lambda_2 + \lambda_3\lambda_4 + (\lambda_1 + \lambda_2)(\lambda_3 + \lambda_4)\right]\lambda^2$$

$$- \left[\lambda_1\lambda_2(\lambda_3 + \lambda_4) + \lambda_3\lambda_4(\lambda_1 + \lambda_2)\right]\lambda + \lambda_1\lambda_2\lambda_3\lambda_4 = 0 \qquad (12)$$

If the coefficients of powers of λ in equation (11a) are equated to the coefficients of like powers of λ in equation (12), four simultaneous linear algebraic equations in the gains are obtained. They are

$$\left.\begin{aligned}
&L_c k_3 + N_c k_4 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 - Y_\beta \\
&-(\alpha L_c - N_c)k_1 - L_c k_2 + Y_\beta L_c k_3 + Y_\beta N_c k_4 = \lambda_1\lambda_2 + \lambda_3\lambda_4 + (\lambda_1 + \lambda_2)(\lambda_3 + \lambda_4) + \alpha L_\beta - N_\beta \\
&\frac{g}{V} L_c k_1 - Y_\beta L_c k_2 + (N_\beta L_c - L_\beta N_c)k_3 + \alpha(N_\beta L_c - L_\beta N_c)k_4 = \lambda_1\lambda_2(\lambda_3 + \lambda_4) + \lambda_3\lambda_4(\lambda_1 + \lambda_2) - \frac{g}{V} L_\beta \\
&(L_\beta N_c - N_\beta L_c)k_2 + \frac{g}{V}(L_\beta N_c - N_\beta L_c)k_4 = \lambda_1\lambda_2\lambda_3\lambda_4
\end{aligned}\right\} \qquad (13)$$

Equations (13) are the desired linear equations for the feedback gains k_i in terms of the parameters of the airplane and the roots λ_i of the characteristic equation. The ratio c_1/c_2 (when $c_1/c_2 = \delta_a/\delta_r$), with either c_1 or c_2 equal to 1, must be specified before these equations can be solved. However, it is clear that this ratio should be chosen equal to the ratio of augmenter deflection limits, which the designer must set from practical considerations. Then, the ratio of the control deflections will have the proper value when the augmenter system is saturated by a large disturbance.

EXAMPLE CALCULATIONS

Consider an airplane having the characteristics and flight conditions given in the following table:

b, ft	22.36
g, ft/sec^2	32.2
I_X, slug-ft^2	5021
I_Z, slug-ft^2	67 199
m, slugs	390.4
q, lb/sq ft	200
S, sq ft	200
V, ft/sec	6000
Altitude, ft	125 000
Mach number	6
α, deg	20
C_{l_β}, per radian	0.015
C_{n_β}, per radian	0.31
C_{Y_β}, per radian	-1.0
$C_{l_{\delta_a}}$, per radian	-0.075
$C_{l_{\delta_r}}$, per radian	-0.15
$C_{n_{\delta_a}}$, per radian	0.08
$C_{n_{\delta_r}}$, per radian	-0.108

The roots of the unaugmented characteristic equation are

$$\lambda_1 = 0$$

$$\lambda_2 = 0.0044904$$

$$\lambda_3 = -0.0107835 + 1.78698j$$

$$\lambda_4 = -0.0107835 - 1.78698j$$

These characteristics are representative of an airplane similar to the X-15 flying at Mach 6 and an altitude of 125 000 feet. The components of the vector $\vec{c}$, are

$$c_1 = 1$$

$$c_2 = \frac{\delta_r}{\delta_a} = \frac{1}{4}$$

The selection of values for c_1 and c_2 is based on the assumed relative authorities of the ailerons and rudder for augmentation purposes. Maximum authorities of the rudder and ailerons should be attained simultaneously for the present choice of $\vec{c}$. The desired stability is represented by the roots λ_i of the controlled system which were chosen to satisfy military handling-qualities specifications (ref. 4) and are

$$\lambda_1 = -0.0346$$

$$\lambda_2 = -0.693$$

$$\lambda_3 = -0.346 - 3.14j$$

$$\lambda_4 = -0.346 + 3.14j$$

where $j = \sqrt{-1}$.

The condition for complete controllability, from equation (A7) in the appendix, is

$$\det\left[[G]\vec{c},\ [A][G]\vec{c},\ [A]^2[G]\vec{c},\ [A]^3[G]\vec{c}\right] \neq 0$$

The inequality is satisfied with the numerical data of the example.

The solution of equation (13), with use of the example data, resulted in the following gains:

$$k_1 = 0.939796$$

$$k_2 = 0.0026195$$

$$k_3 = 0.0713637$$

$$k_4 = 0.0391162$$

A similar set of gains can be computed for any set of desired roots λ_i of the characteristic equation (eq. (9)), and, in a complete analysis, the roots may have to be chosen selectively in order to insure reasonable gains.

Figure 1 presents the motions of the controlled airplane, for the particular set of chosen roots, following an initial sideslip angle of 10.9°. The period of the oscillatory motion is 2 seconds and the time to damp to one-half amplitude is 2 seconds. The motions of the free airplane system are included in figure 1 for comparison. The free airplane has a period of about 3.5 seconds, and the time to damp to one-half amplitude is about 64 seconds. Although the motions were computed for a large initial sideslip angle, the aileron and rudder deflections are within probable limits for an airplane of the type considered.

CONCLUDING REMARKS

State-vector control theory has been applied to a stability problem of high-performance aircraft. Analytical expressions have been derived for feedback gains that will produce any desired roots of the characteristic equation of the augmented system. The gains are given in terms of the parameters of the system and the desired roots. A numerical example illustrates the practicability of the method.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., April 7, 1965.

APPENDIX

COMPLETE CONTROLLABILITY

A linear dynamical system is said to be completely controllable if, at any initial time t, any initial state x can be taken to the origin in a finite length of time by the application of a suitable control function.

Consider the mathematical model of a linear dynamical system

$$\frac{d\vec{x}}{dt} = [A]\vec{x} + [G]\vec{u} \tag{A1}$$

where $\vec{x}$ is an n-vector, the state of the system, and $\vec{u}$ is an m-vector, the control of the system. The rectangular matrices $[A]$ and $[G]$ are, in general, functions of time. However, for present purposes, they are assumed to be constant.

A necessary and sufficient condition for the constant system (eq. (A1)) to be completely controllable is

$$\text{rank}\Big[[G],\ [A][G],\ [A]^2[G],\ .\ .\ .\ ,\ [A]^{n-1}[G]\Big] = n \tag{A2}$$

Suppose that the control vector $\vec{u}$ is a scalar s times a known vector $\vec{c}$, or

$$\vec{u} \equiv \vec{c}s \tag{A3}$$

Define a new n-vector

$$\vec{w} \equiv [G]\vec{c} \tag{A4}$$

so that the dynamical system is represented by

$$\frac{d\vec{x}}{dt} = [A]\vec{x} + \vec{w}s \tag{A5}$$

The condition for complete controllability of equation (A5) is

$$\text{rank}[M] \equiv \text{rank}\Big[\vec{w},\ [A]\vec{w},\ [A]^2\vec{w},\ .\ .\ .\ ,\ [A]^{n-1}\vec{w}\Big] = n \tag{A6}$$

or, since $[M]$ is an $n \times n$ matrix,

$$\det\left|\vec{w},\ [A]\vec{w},\ [A]^{2}\vec{w},\ .\ .\ .\ ,\ [A]^{n-1}\vec{w}\right| \neq 0 \tag{A7}$$

A more detailed and general discussion of complete controllability is presented in reference 3.

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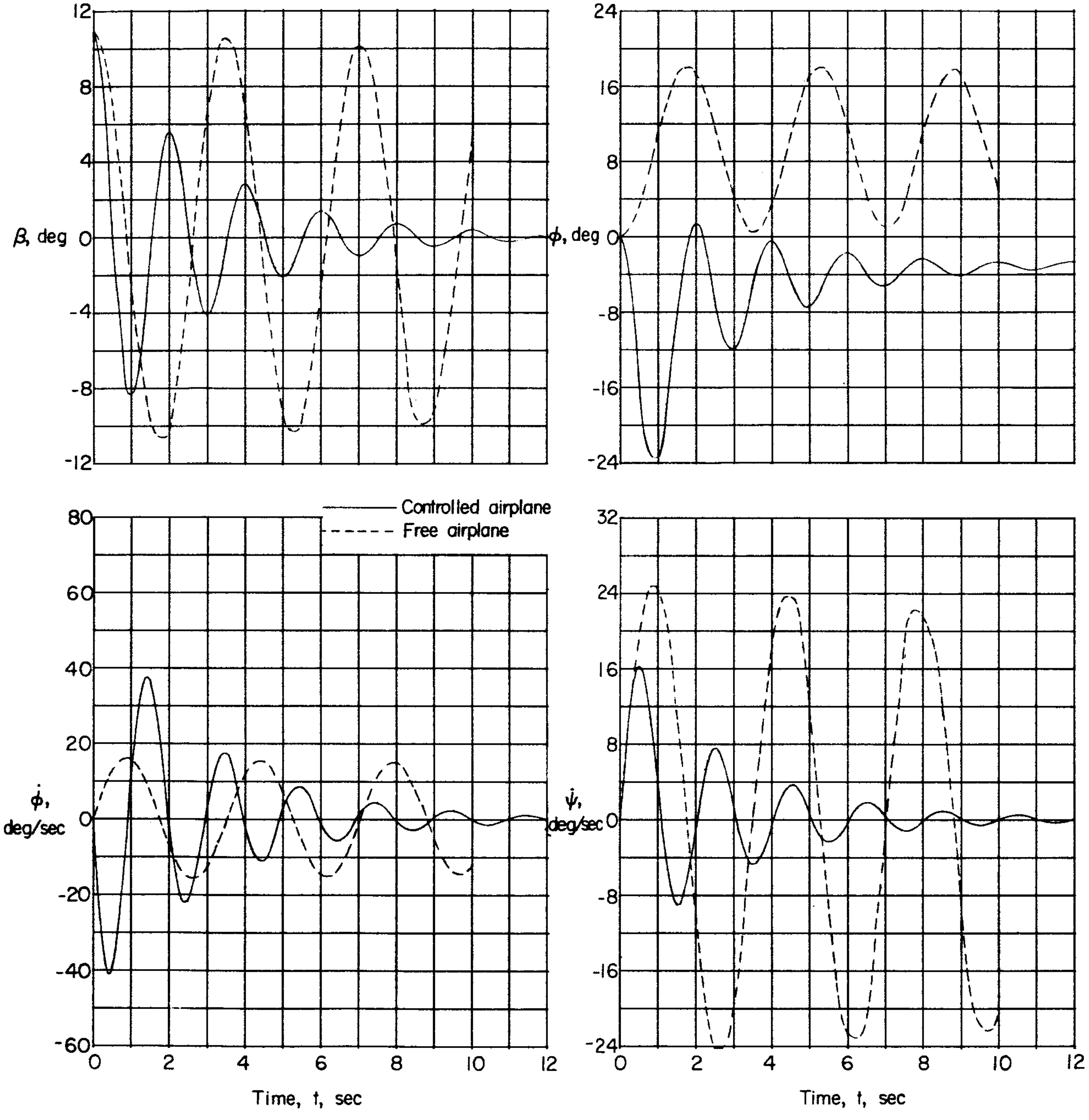

Figure 1.- Motion of airplane and controls.

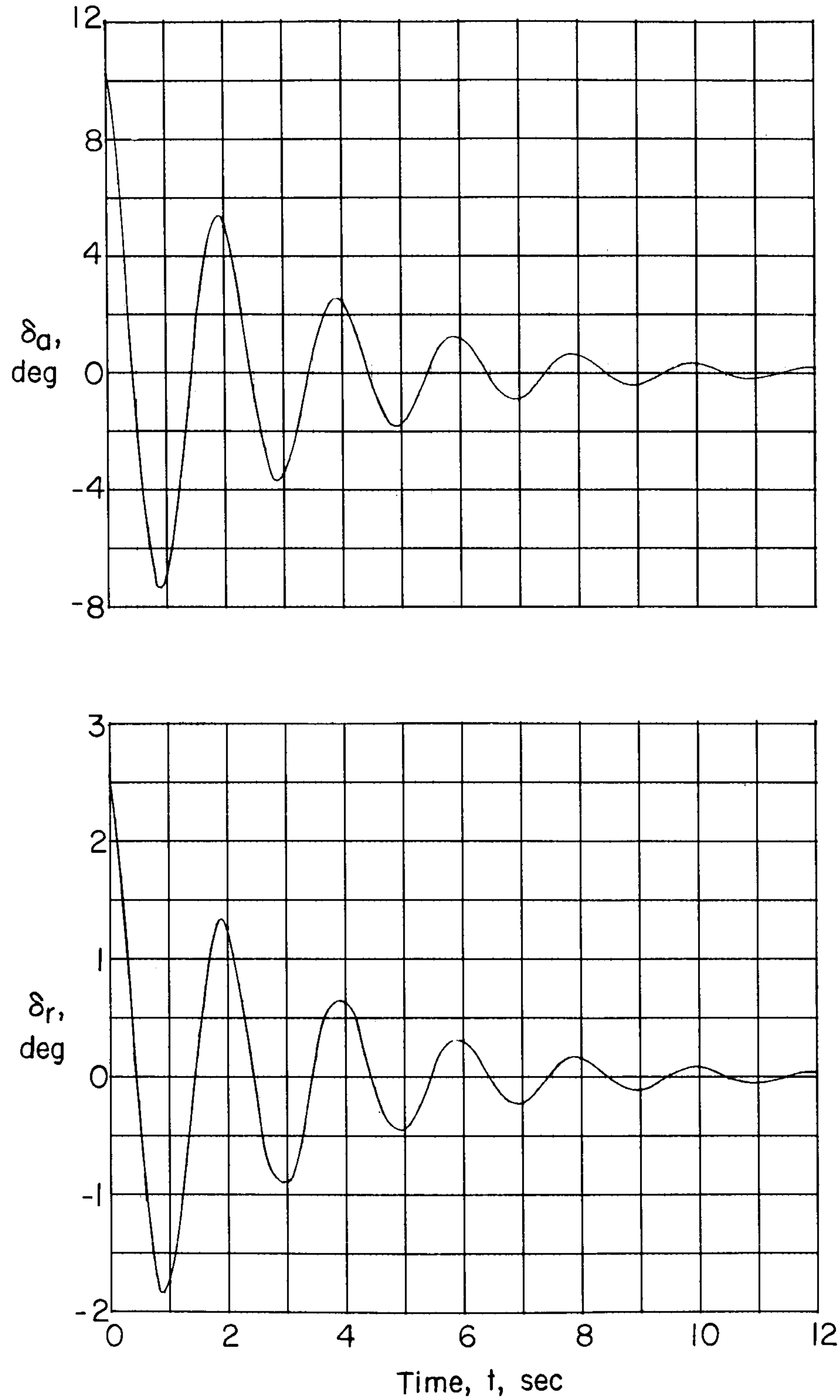

Figure 1.- Concluded.

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